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ADDENDUM TO “COMMENSURABILITY BETWEEN ONCE-PUNCTURED TORUS GROUPS AND ONCE-PUNCTURED KLEIN BOTTLE GROUPS”

MIKIO FUROKAWA

1. INTRODUCTION

The main purpose of this addendum to [4] is to present a proof to [4, Proposition 3.7] which gives a classification of elliptic generator triples of the fundamental group of the quotient orbifold of the once-punctured Klein bottle (see Definition 2.1 and Proposition 2.2). We also prove the “converse” of [4, Theorem 5.1], namely, we give a condition for a faithful type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representation of the fundamental group of the once-punctured torus to be “commensurable” with that of the once-punctured Klein bottle by using Proposition 3.7 and Theorem 5.1 in the original paper (see Definitions 3.1, 3.2 and Theorem 3.13).

The rest of this paper is organized as follows. In Section 2, we give a proof to [4, Proposition 3.7] (see Proposition 2.2). In Section 3, we prove the “converse” of [4, Theorem 5.1] (see Theorem 3.13).

2. CLASSIFICATION OF ELLIPTIC GENERATOR TRIPLES

In this section, we give a proof to [4, Proposition 3.7]. To this end, we first introduce some notations and recall the definition of elliptic generators.

Let $N_{2,1}$ be the once-punctured Klein bottle and let $\iota_{N_{2,1}} : N_{2,1} \to N_{2,1}$ be the involution illustrated in Figure 1. Then we denote the quotient orbifold $N_{2,1}/\iota_{N_{2,1}}$ by $\mathcal{O}_{N_{2,1}}$ and denote the covering projection from $N_{2,1}$ to $\mathcal{O}_{N_{2,1}}$ by $p_{N_{2,1}}$. We identify $\pi_1(N_{2,1})$ with the image of the inclusion $\pi_1(N_{2,1}) \to \pi_1(\mathcal{O}_{N_{2,1}})$ induced by the projection $p_{N_{2,1}}$. Then $\pi_1(N_{2,1})$ is regarded as a normal subgroup of $\pi_1(\mathcal{O}_{N_{2,1}})$ of index 2,

$$\pi_1(N_{2,1}) = \langle Y_1, Y_2 \mid - \rangle \triangleleft \pi_1(\mathcal{O}_{N_{2,1}}) = \langle Q_0, Q_1, Q_2 \mid Q_0^2 = Q_1^2 = Q_2^2 = 1 \rangle,$$

such that $Y_1 = Q_0Q_1$ and $Y_2 = Q_0Q_2$. Set $K_{N_{2,1}} = (Y_1Y_2Y_1^{-1}Y_2)^{-1}$, $K_0 = Q_1^{Q_0}$ and $K_2 = Q_1^{Q_2}$, where $A^B = BAB^{-1}$. Then $K_{N_{2,1}}$ is represented by the puncture of $N_{2,1}$, and K_0 and K_2 are represented by the reflector lines which generate the corner reflector of order ∞. By the identification, we also obtain $K_{N_{2,1}} = K_2K_0$.

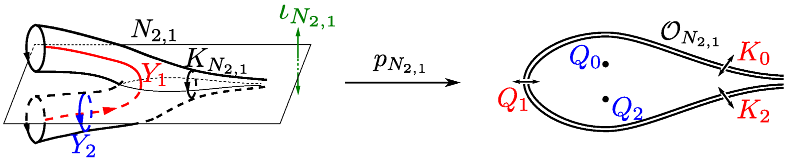

FIGURE 1. The involution $\iota_{N_{2,1}}$ of $N_{2,1}$

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Definition 2.1. An ordered triple (Q_0, Q_1, Q_2) of elements of $\pi_1(\mathcal{O}_{N_{2,1}})$ is called an *elliptic generator triple* of $\pi_1(\mathcal{O}_{N_{2,1}})$ if its members generate $\pi_1(\mathcal{O}_{N_{2,1}})$ and satisfy $Q_0^2 = Q_1^2 = Q_2^2 = 1$ and ${Q_1}^{Q_2}{Q_1}^{Q_0} = K_2K_0$. A member of an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$ is called an *elliptic generator* of $\pi_1(\mathcal{O}_{N_{2,1}})$.

Now we introduce Proposition 3.7 in the original paper.

Proposition 2.2. *The elliptic generator triples of $\pi_1(\mathcal{O}_{N_{2,1}})$ are characterized as follows.*

(1) *For any elliptic generator triple (Q_0, Q_1, Q_2) of $\pi_1(\mathcal{O}_{N_{2,1}})$, the following hold:*

(1.1) *The triples in the following bi-infinite sequence are also elliptic generator triples of $\pi_1(\mathcal{O}_{N_{2,1}})$.*

$$\dots, ({Q_0}^{K_0K_2}, {Q_1}^{K_0K_2}, {Q_2}^{K_0K_2}),\ ({Q_2}^{K_0}, {Q_1}^{K_0}, {Q_0}^{K_0}),\ (Q_0, Q_1, Q_2),$$
$$({Q_2}^{K_2}, {Q_1}^{K_2}, {Q_0}^{K_2}),\ ({Q_0}^{K_2K_0}, {Q_1}^{K_2K_0}, {Q_2}^{K_2K_0}),\ \dots$$

To be precise, the following holds. Let $\{Q_j\}$ be the sequence of elements of $\pi_1(\mathcal{O}_{N_{2,1}})$ obtained from (Q_0, Q_1, Q_2) by applying the following rule:

$$Q_j^{K_0} = Q_{-j-1},\ Q_j^{K_2} = Q_{-j+5}.$$

Then the triple $(Q_{3k}, Q_{3k+1}, Q_{3k+2})$ is also an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$ for any $k \in \mathbb{Z}$.

(1.2) *$(Q_2, {Q_1}^{Q_2Q_0}, {Q_0}^{Q_2})$ is also an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$.*

(2) *Conversely, any elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$ is obtained from a given elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$ by successively applying the operations in (1).*

To prove Proposition 2.2, we need to introduce some definitions and notations. By a *word* in $\{Q_0, Q_1, Q_2\}$, we mean a finite sequence $Q_{i_1}Q_{i_2}\dots Q_{i_t}$ where $Q_{i_k} \in \{Q_0, Q_1, Q_2\}$. Here we call Q_{i_k} the *k-th letter* of the word. In particular, the first letter Q_{i_1} of the word is called the *initial letter* of the word and the last letter Q_{i_t} of the word is called the *terminal letter* of the word. The *inverse* of a word $V = Q_{i_1}Q_{i_2}\dots Q_{i_t}$ in $\{Q_0, Q_1, Q_2\}$ is the word $V^{-1} = Q_{i_t}Q_{i_{t-1}}\dots Q_{i_1}$. The word length of V is denoted by $l(V)$. A word $V = Q_{i_1}Q_{i_2}\dots Q_{i_t}$ in $\{Q_0, Q_1, Q_2\}$ is *reduced* if $Q_{i_k} \neq Q_{i_{k+1}}$ for any $k = 1, \dots, t-1$. Note that any element in $\pi_1(\mathcal{O}_{N_{2,1}})$ is uniquely represented by a reduced word. For two words U, V in $\{Q_0, Q_1, Q_2\}$, by $U \equiv V$ we denote the *visual equality* of U and V, meaning that if $U = Q_{i_1}Q_{i_2}\dots Q_{i_t}$ and $V = Q_{j_1}Q_{j_2}\dots Q_{j_u}$ $(Q_{i_k}, Q_{j_l} \in \{Q_0, Q_1, Q_2\})$, then $t = u$ and $Q_{i_k} = Q_{j_k}$ for each $k = 1, \dots, t$. For example, two words $Q_0Q_1Q_1Q_2$ and Q_0Q_2 are *not* visually equal, though $Q_0Q_1Q_1Q_2$ and Q_0Q_2 are equal as elements of $\pi_1(\mathcal{O}_{N_{2,1}})$.

Proof of Proposition 2.2. The author got the idea of the proof from the proof of [2, Proposition 10.7] and [1, Lemma 2.1.7].

Since (1) can be proved by direct calculation, we give a proof of (2). For a given elliptic generator triple (Q_0, Q_1, Q_2), set $K_0 = Q_1^{Q_0}$ and $K_2 = Q_1^{Q_2}$, and let τ and σ be the automorphism of $\pi_1(\mathcal{O})$ defined by

$$(\tau(Q_0), \tau(Q_1), \tau(Q_2)) = (Q_2^{K_2}, Q_1^{K_2}, Q_0^{K_2}),$$
$$(\sigma(Q_0), \sigma(Q_1), \sigma(Q_2)) = (Q_2, Q_1^{Q_2Q_0}, Q_0^{Q_2}).$$

Then τ and σ preserve $K_{N_{2,1}}$ and hence they map elliptic generator triples to elliptic generator triples. Moreover, the operations in (1.1) is given by τ^n, and the operation in (1.2) is given by σ. Hence we have only to show the following lemma.

Lemma 2.3. *The group of automorphisms of $\pi_1(\mathcal{O}_{N_{2,1}})$ preserving $K_{N_{2,1}}$ is generated by σ and τ.*

To prove this lemma, we prepare two claims.

Claim 2.4. *Let f be an automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ which preserves $K_{N_{2,1}}$. Then for each $j = 0, 2$, we have*

$$f(K_j) = K_{j'}^{K^n_{N_{2,1}}} \qquad \text{for some } n \in \mathbb{Z} \text{ and some } j' \in \{0, 2\}.$$

Proof of Claim 2.4. We first note that $\pi_1(\mathcal{O}_{N_{2,1}})$ is regarded as a subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. Then $\langle K_0, K_2\rangle$ is regarded as the stabilizer of ∞ and $K_{N_{2,1}} = K_2K_0$ is regarded as a parabolic transformation $K_{N_{2,1}}(z) = z + 2$. On the other hand, since $f(K_2)f(K_0) = K_2K_0 = K_{N_{2,1}}$, we see that

$$f(K_0)K_{N_{2,1}}(f(K_0))^{-1} = f(K_0)f(K_2)f(K_0)(f(K_0))^{-1} = f(K_0)f(K_2) = K^{-1}_{N_{2,1}}.$$

This implies that $f(K_0)K_{N_{2,1}}(f(K_0))^{-1}$ is parabolic and that $\mathrm{Fix}(f(K_0)K_{N_{2,1}}(f(K_0))^{-1}) = \{\infty\}$, where $\mathrm{Fix}(A)$ denotes the fixed point set of A in $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. By $\mathrm{Fix}(K_{N_{2,1}}) = \{\infty\}$ and $\mathrm{Fix}(f(K_0)K_{N_{2,1}}(f(K_0))^{-1}) = f(K_0)(\mathrm{Fix}(K_{N_{2,1}}))$, we have $f(K_0)(\infty) = \infty$. Hence $f(K_0) \in \langle K_0, K_2\rangle$ and therefore $f(K_0) = K_{j'}^{K^n_{N_{2,1}}}$ for some $n \in \mathbb{Z}$ and some $j' \in \{0, 2\}$. By a similar argument, we obtain the desired result for $f(K_2)$. $\square$

Claim 2.5. *Let f be an automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ such that $f(K_j) = K_j$ for each $j = 0, 2$. Suppose that $f(Q_s) = W_sQ_sW_s^{-1}$ for each $s = 0, 1, 2$, where W_s is a reduced word in $\{Q_0, Q_1, Q_2\}$ whose terminal letter is different from Q_s. Then the following hold.*

(1) *If W_1 is a trivial word, then W_j is also a trivial word for each $j = 0, 2$.*

(2) *If W_1 is a non-trivial word, then one of the following holds for each $j = 0, 2$.*
  - (i) *$W_1Q_1Q_j \equiv W_jQ_jW_j^{-1}$. In particular, the initial letter of W_1 is Q_j.*
  - (ii) *$W_1 \equiv W_jQ_jW_j^{-1}Q_j$. In particular, the terminal letter of W_1 is Q_j.*
  - (iii) *$W_1Q_j \equiv W_jQ_jW_j^{-1}$. In particular, the terminal letter of W_1 is different from Q_j.*

Proof of Claim 2.5. For each $j = 0, 2$, we have the following identity:

$$Q_jQ_1Q_j = K_j = f(K_j) = f(Q_jQ_1Q_j) = W_jQ_jW_j^{-1} \cdot W_1Q_1W_1^{-1} \cdot W_jQ_jW_j^{-1}.$$

This implies that $Q_j \cdot W_jQ_jW_j^{-1} \cdot W_1$ commutes with Q_1. Since $\pi_1(\mathcal{O}_{N_{2,1}})$ is isomorphic to the free product of three cyclic groups $\langle Q_s\rangle$ of order 2, we have

$$\text{(Eq1)} \qquad Q_j \cdot W_jQ_jW_j^{-1} \cdot W_1 = Q_1 \text{ or } 1.$$

To show the assertion (1), we assume that W_1 is a trivial word. Then, by the identity (Eq1), we have $Q_j \cdot W_jQ_jW_j^{-1} = Q_1$ or 1 . By the abelianization of this identity, we have $Q_j \cdot W_jQ_jW_j^{-1} = 1$. This implies that W_j commutes with Q_j, and hence $W_j = Q_j$ or 1. Since the terminal letter of W_j is different from Q_j, we have $W_j = 1$. So we obtain the desired result.

Next, we show the assertion (2). If either W_0 or W_2 is a trivial word, then the identity (Eq1) implies that $W_1 = Q_1$ or 1. This is a contradiction. Hence W_j is also a non-trivial word for any $j = 0, 2$.

Suppose first that $Q_j \cdot W_j$ is a reduced word. Then $Q_j \cdot W_jQ_jW_j^{-1}$ is also a reduced word. Hence the identity (Eq1) implies that the word $Q_j \cdot W_jQ_jW_j^{-1}$, except possibly for the first letter Q_j, is cancelled out by the word W_1, and therefore one of the following holds.

- $W_1 \equiv W_j Q_j W_j^{-1} Q_j Q_1$,
- $W_1 \equiv W_j Q_j W_j^{-1}$ and $Q_j = Q_1$,
- $W_1 \equiv W_j Q_j W_j^{-1} Q_j$.

However, the first identity can not hold because the terminal letter of W_1 is different from Q_1 by the assumption, and second identity can not hold because $j = 0, 2$. Hence the third identity holds. So we obtain the identity in the condition (ii).

Suppose next that $Q_j \cdot W_j$ is not a reduced word, i.e., $W_j \equiv Q_j \cdot V_j$ for some reduced word V_j. Then, by the identity (Eq1), we have

$$\text{(Eq2)} \qquad V_j Q_j W_j^{-1} \cdot W_1 = Q_1 \text{ or } 1.$$

Since $V_j Q_j W_j^{-1}$ is a reduced word, it must be cancelled out by W_1, except possibly for the initial letter of V_j, and therefore one of the following hold.

- $W_1 \equiv W_j Q_j V_j^{-1} Q_1$,
- $W_1 \equiv W_j Q_j V_j'^{-1}$ and $V_j \equiv Q_1 V_j'$ for some reduced word V_j'.
- $W_1 \equiv W_j Q_j V_j^{-1}$.

The first identity can not hold by the fact that the terminal letter of W_1 is different from Q_1. If the second identity or the third identity holds, then the condition (i) or (iii) holds accordingly. □

We now begin to prove Lemma 2.3 by using the above claims.

Let f be an automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ preserving $K_{N_{2,1}}$.

Step 1. For each $j = 0, 2$, we show that we may assume $f(K_j) = K_j$ by post composing a power of τ to f if necessary. By Claim 2.4, we have $f(K_0) = K_{j'}^{K_{N_{2,1}}^n}$ for some $n \in \mathbb{Z}$ and for some $j' \in \{0, 2\}$. Since τ^2 is an inner-automorphism by $K_{N_{2,1}}$, we may assume $f(K_0) = K_{j'}$ by post composing a power of τ^2 to f if necessary. By the assumption $f(K_2)f(K_0) = f(K_{N_{2,1}}) = K_{N_{2,1}}$, we have $f(K_2) = K_{N_{2,1}} f(K_0) = K_2 K_0 f(K_0)$. Hence

$$f(K_2) = K_2 K_0 K_{j'} = \begin{cases} K_2 & \text{if } j' = 0, \\ K_0^{K_2} & \text{if } j' = 2. \end{cases}$$

Since τ maps (K_0, K_2) to $(K_2^{K_0}, K_0)$, we may assume $f(K_j) = K_j$ for each $j = 0, 2$ by post composing τ to f if necessary.

Step 2. For each $s = 0, 1, 2$, we show that we may assume $f(Q_s) = W_s Q_s W_s^{-1}$ by post composing σ to f if necessary. Since $f(Q_s)$ has order 2 and since $\pi_1(\mathcal{O}_{N_{2,1}})$ is isomorphic to the free product of three cyclic groups $\langle Q_s \rangle$ of order 2, we have $f(Q_s) = V_s Q_{\theta(s)} V_s^{-1}$ for some $\theta(s) \in \{0, 1, 2\}$, where V_s is a reduced word whose terminal letter is different from $Q_{\theta(s)}$. By the abelianization of the identity

$$Q_2 Q_1 Q_2 = K_2 = f(K_2) = f(Q_2 Q_1 Q_2) = f(Q_2) f(Q_1) f(Q_2),$$

we have $\theta(1) = 1$. By Step1, we have the following identities:

$$Q_0 Q_1 Q_0 = K_0 = f(K_0) = f(Q_0) f(Q_1) f(Q_0),$$
$$Q_2 Q_1 Q_2 = K_2 = f(K_2) = f(Q_2) f(Q_1) f(Q_2).$$

By these identities, we have the following identity:

$$Q_1 \cdot Q_2 f(Q_2) f(Q_0) Q_0 = Q_2 f(Q_2) f(Q_0) Q_0 \cdot Q_1.$$

This implies that $Q_2 f(Q_2) f(Q_0) Q_0 = Q_2 V_2 Q_{\theta(2)} V_2^{-1} V_0 Q_{\theta(0)} V_0^{-1} Q_0$ commutes with Q_1. As in the proof of Claim 2.5, we see that

$$Q_2 V_2 Q_{\theta(2)} V_2^{-1} V_0 Q_{\theta(0)} V_0^{-1} Q_0 = 1 \text{ or } Q_1.$$

Since the word length of the left hand side of the above identity is even, we have $Q_2 V_2 Q_{\theta(2)} V_2^{-1} V_0 Q_{\theta(0)} V_0^{-1} Q_0 = 1$. By the abelianization of this identity, we have

$$Q_2 Q_{\theta(2)} Q_{\theta(0)} Q_0 = 1.$$

This implies that $\theta(0), \theta(2) \in \{0, 2\}$ and $\theta(0) \neq \theta(2)$. Hence θ must be a permutation on the set $\{0, 1, 2\}$ such that $\theta(1) = 1$. Since σ preserves K_0 and K_2 and since σ maps (Q_0, Q_1, Q_2) to $(Q_2, Q_1^{Q_2 Q_0}, Q_0^{Q_2})$, we may assume $\theta = id$ by post composing σ to f if necessary. Hence $f(Q_s) = W_s Q_s W_s^{-1}$ for each $s = 0, 1, 2$, where W_s is a reduced word whose terminal letter is different from Q_s.

Step 3. We show that $f = (\sigma^2)^{n+1}$. If W_1 is a trivial word, W_j is a trivial word for any $j = 0, 2$ by Claim 2.5, and therefore $f = id$. So we assume that W_1 is a non-trivial word. Since the terminal letter of W_1 is different from Q_1, we assume that the terminal letter of W_1 is Q_0. (The other case is treated by a parallel argument.) Then the condition (2)-(i) or (2)-(ii) in Claim 2.5 holds for $j = 0$, and the condition (2)-(i) or (2)-(iii) in Claim 2.5 holds for $j = 2$. Note that the number of Q_1 contained W_1 is odd or even according to whether the condition (2)-(i) in Claim 2.5 holds or not. If the number of Q_1 contained W_1 is odd, then the condition (2)-(i) in Claim 2.5 holds for each $j = 0, 2$. In particular, the initial letter of W_1 is Q_0 and Q_2, a contradiction. Hence the number of Q_1 contained W_1 is even. Then the condition (2)-(ii) in Claim 2.5 holds for $j = 0$, and the condition (2)-(iii) in Claim 2.5 holds for $j = 2$, namely, we have $W_1 \equiv W_0 Q_0 W_0^{-1} Q_0$ and $W_1 Q_2 \equiv W_2 Q_2 W_2^{-1}$. Thus we see $W_0 Q_0 W_0^{-1} Q_0 Q_2 \equiv W_1 Q_2 \equiv W_2 Q_2 W_2^{-1}$. This implies that i-th letter of $W_0 Q_0 W_0^{-1} Q_0 Q_2$ is equal to $(l - i + 1)$-th letter of $W_0 Q_0 W_0^{-1} Q_0 Q_2$, where $l = l(W_0 Q_0 W_0^{-1} Q_0 Q_2)$. Hence $W_0 \equiv (Q_2 Q_0)^n Q_2$ for some $n \in \mathbb{N}$, and therefore $W_1 \equiv (Q_2 Q_0)^{2(n+1)}$ and $W_2 \equiv (Q_2 Q_0)^{n+1}$. Thus we see

$$f(Q_0) = Q_0^{(Q_2 Q_0)^{n+1}}, \quad f(Q_1) = Q_1^{(Q_2 Q_0)^{2(n+1)}} \quad \text{and} \quad f(Q_2) = Q_2^{(Q_2 Q_0)^{n+1}}.$$

On the other hand, $(\sigma^2(Q_0), \sigma^2(Q_1), \sigma^2(Q_2)) = (Q_0^{Q_2 Q_0}, Q_1^{(Q_2 Q_0)^2}, Q_2^{Q_2 Q_0})$. Thus we have $f = (\sigma^2)^{n+1}$. Hence we obtain the desired result. □

Remark 2.6. It should be noted that the proof of Proposition 2.2 does not use the condition that $(f(Q_0), f(Q_1), f(Q_2))$ generates $\pi_1(\mathcal{O}_{N_{2,1}})$. Hence, in Definition 2.1, the condition that members of the triple generate $\pi_1(\mathcal{O}_{N_{2,1}})$ is actually a consequence of the other conditions (cf. [4, Remark 3.6]).

Definition 2.7. For an elliptic generator triple (Q_0, Q_1, Q_2) of $\pi_1(\mathcal{O}_{N_{2,1}})$, the bi-infinite sequence $\{Q_j\}$ in Proposition 2.2(1.1) is called the *sequence of elliptic generators* of $\pi_1(\mathcal{O}_{N_{2,1}})$ (associated with (Q_0, Q_1, Q_2)).

In preparation for the next section, we recall the definition of elliptic generators of the fundamental group of the quotient orbifold of the once-punctured torus.

Let $\Sigma_{1,1}$ be the once-punctured torus and let $\iota_{\Sigma_{1,1}} : \Sigma_{1,1} \to \Sigma_{1,1}$ be the involution illustrated in Figure 2. Then we denote the quotient orbifold $\Sigma_{1,1}/\iota_{\Sigma_{1,1}}$ by $\mathcal{O}_{\Sigma_{1,1}}$ and denote the covering projection from $\Sigma_{1,1}$ to $\mathcal{O}_{\Sigma_{1,1}}$ by $p_{\Sigma_{1,1}}$. We identify $\pi_1(\Sigma_{1,1})$ with the image of the inclusion $\pi_1(\Sigma_{1,1}) \to \pi_1(\mathcal{O}_{\Sigma_{1,1}})$ induced by the projection $p_{\Sigma_{1,1}}$. Then $\pi_1(\Sigma_{1,1})$ is regarded as a normal subgroup of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ of index 2,

$$\pi_1(\Sigma_{1,1}) = \langle X_1, X_2 \mid - \rangle \lhd \pi_1(\mathcal{O}_{\Sigma_{1,1}}) = \langle P_0, P_1, P_2 \mid P_0^2 = P_1^2 = P_2^2 = 1 \rangle,$$

such that $X_1 = P_2P_1$ and $X_2 = P_0P_1$. Set $K_{\Sigma_{1,1}} = [X_1, X_2] = X_1X_2X_1^{-1}X_2^{-1}$, $K = (P_0P_1P_2)^{-1}$. Then $K_{\Sigma_{1,1}}$ and K are represented by the punctures of $\Sigma_{1,1}$ and $\mathcal{O}_{\Sigma_{1,1}}$, respectively.

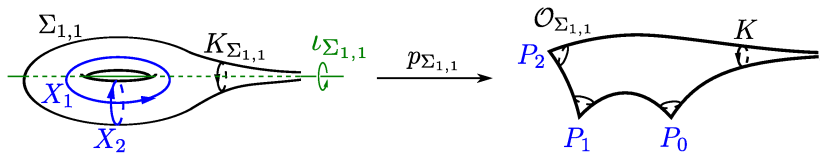

FIGURE 2. The involution $\iota_{\Sigma_{1,1}}$ of $\Sigma_{1,1}$

Definition 2.8. An ordered triple (P_0, P_1, P_2) of elements of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ is called an *elliptic generator triple* of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ if its members generate $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and satisfy $P_0^2 = P_1^2 = P_2^2 = 1$ and $(P_0P_1P_2)^{-1} = K$. A member of an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ is called an *elliptic generator* of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$.

Definition 2.9. For an elliptic generator triple (P_0, P_1, P_2) of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$, let $\{P_j\}$ be the bi-infinite sequence defined as follows (see [1, Proposition 2.1.6(1.1)] and [4, Proposition 3.3(1.1)]).

$$\dots, P_2^{K^{-2}}, P_0^{K^{-1}}, P_1^{K^{-1}}, P_2^{K^{-1}}, P_0, P_1, P_2, P_0^{K}, P_1^{K}, P_2^{K}, P_0^{K^2}, \dots$$

We call the sequence $\{P_j\}$ the *sequence of elliptic generators* of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ (associated with (P_0, P_1, P_2)).

3. COMMENSURABILITY

In this section, we prove the "converse" of [4, Theorem 5.1], namely, we give a condition for a faithful type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\Sigma_{1,1})$ to be commensurable with that of $\pi_1(N_{2,1})$. We first introduce some notations and facts.

Let $\Sigma_{1,2}$, $\mathcal{O}_{\Sigma_{1,2}}$, $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ be the twice-punctured torus, the $(2,2,2,\infty)$-orbifold (i.e., the orbifold with underlying space a punctured sphere and with four cone points of cone angle π), the $(2;2,\infty]$-orbifold (i.e., the orbifold with underlying space a disk and with a cone point of cone angle π and with a corner reflector of order 2 and a corner reflector of order ∞) and the $[2,2,2,\infty]$-orbifold (i.e., the orbifold with underlying space a disk and with three corner reflectors of order 2 and a corner reflector of order ∞), respectively. Note that $\mathcal{O}_{\Sigma_{1,2}}$ is a quotient orbifold of $\Sigma_{1,2}$ by an involution and that both $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ are common quotient orbifolds of $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{N_{2,1}}$ by involutions (see [4, Section2] for details). Their (orbifold) fundamental groups have the following presentations:

$$\pi_1(\Sigma_{1,2}) = \langle Z_1, Z_2, Z_3 \mid - \rangle,$$
$$\pi_1(\mathcal{O}_{\Sigma_{1,2}}) = \langle R_0, R_1, R_2, R_3 \mid R_0^2 = R_1^2 = R_2^2 = R_3^2 = 1 \rangle,$$
$$\pi_1(\mathcal{O}_\alpha) = \langle S_0, S_1, S_2 \mid S_0^2 = S_1^2 = S_2^2 = 1, (S_1S_2)^2 = 1 \rangle,$$
$$\pi_1(\mathcal{O}_\beta) = \left\langle T_0, T_1, T_2, T_3 \;\middle|\; \begin{array}{c} T_0^2 = T_1^2 = T_2^2 = T_3^2 = 1, \\ (T_0T_1)^2 = (T_1T_2)^2 = (T_2T_3)^2 = 1 \end{array} \right\rangle.$$

Here the generators satisfy the following conditions:

$$Z_1 = R_0R_1,\ Z_2 = R_2R_1,\ Z_3 = R_1R_3,\ K_{\Sigma_{1,2}} = K_{\mathcal{O}_{\Sigma_{1,2}}},\ K'_{\Sigma_{1,2}} = (K^{-1}_{\mathcal{O}_{\Sigma_{1,2}}})^{R_3},$$
$$P_0 = S_0^{S_2},\ P_1 = S_1S_2,\ P_2 = S_0,$$
$$Q_0 = S_0^{S_2},\ Q_1 = S_1,\ Q_2 = S_0,$$
$$P_0 = T_0T_1,\ P_1 = T_1T_2,\ P_2 = T_2T_3,$$
$$Q_0 = T_1T_2,\ Q_1 = T_3^{T_1},\ Q_2 = T_0T_1,$$

where $K_{\Sigma_{1,2}} = Z_1Z_2Z_3$, $K'_{\Sigma_{1,2}} = Z_2Z_1Z_3$ and $K_{\mathcal{O}_{\Sigma_{1,2}}} = R_0R_1R_2R_3$, which are represented by the punctures of $\Sigma_{1,2}$ and $\mathcal{O}_{\Sigma_{1,2}}$ (see Figure 3).

In summary, we have the commutative diagram of double coverings as shown in Figure 3. Every arrow represents a double covering (see [4, Section2] for details).

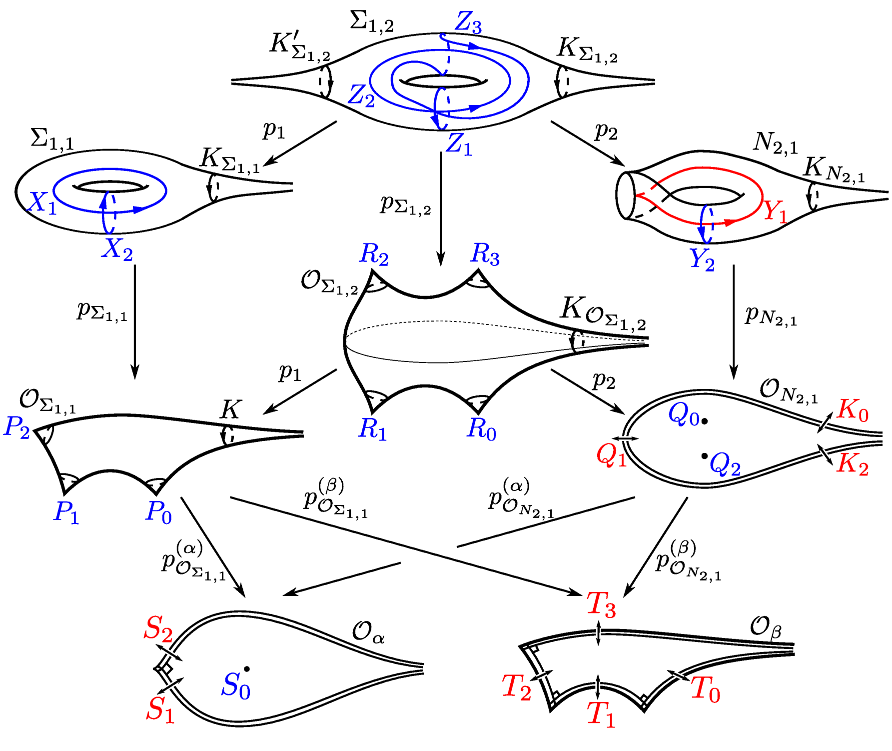

FIGURE 3

Definition 3.1. (1) For $F = \Sigma_{1,1}$, $N_{2,1}$ $\Sigma_{1,2}$, $\mathcal{O}_{\Sigma_{1,1}}$, $\mathcal{O}_{N_{2,1}}$, $\mathcal{O}_{\Sigma_{1,2}}$, $\mathcal{O}_\alpha$ or $\mathcal{O}_\beta$, a representation $\rho : \pi_1(F) \to \mathrm{PSL}(2, \mathbb{C})$ is *type-preserving* if it is irreducible (equivalently, it does not have a common fixed point in $\partial\mathbb{H}^3$) and sends peripheral elements to parabolic transformations.

(2) Type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representations ρ and ρ' are *equivalent* if $i_g \circ \rho = \rho'$, where i_g is the inner automorphism, $i_g(h) = ghg^{-1}$, of $\mathrm{PSL}(2,\mathbb{C})$ determined by g.

In the above definition, if F is an orbifold with reflector lines, an element of $\pi_1(F)$ is said to be *peripheral* if it is (the image of) a peripheral element of $\pi_1(\tilde{F})$, where $\tilde{F}$ is the orientation double covering of F.

Definition 3.2. Let ρ_1 and ρ_2 be type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representations of $\pi_1(\Sigma_{1,1})$ (resp. $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$) and $\pi_1(N_{2,1})$ (resp. $\pi_1(\mathcal{O}_{N_{2,1}})$), respectively. The representations ρ_1 and ρ_2 are *commensurable* if there exist a double covering p_1 from $\Sigma_{1,2}$ (resp. $\mathcal{O}_{\Sigma_{1,2}}$) to $\Sigma_{1,1}$ (resp. $\mathcal{O}_{\Sigma_{1,1}}$) and a double covering p_2 from $\Sigma_{1,2}$ (resp. $\mathcal{O}_{\Sigma_{1,2}}$) to $N_{2,1}$ (resp. $\mathcal{O}_{N_{2,1}}$) such that $\rho_1 \circ (p_1)_*$ and $\rho_2 \circ (p_2)_*$ are equivalent, namely $\rho_1 \circ (p_1)_* = i_g \circ \rho_2 \circ (p_2)_*$ for some $g \in \mathrm{PSL}(2,\mathbb{C})$. After replacing ρ_2 with $i_g \circ \rho_2$, without changing the equivalence class, the last identity can be replaced with the identity $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$.

In this paper, we study the following problem which is a "converse" of [4, Problem 2.3].

Problem 3.3. For a given type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representation ρ_1 of $\pi_1(\Sigma_{1,1})$ (resp. $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$), when does there exist a type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representation ρ_2 of $\pi_1(N_{2,1})$ (resp. $\pi_1(\mathcal{O}_{N_{2,1}})$) which is commensurable with ρ_1?

To answer this problem, we recall the definitions of complex probabilities of type-preserving representations of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and $\pi_1(\mathcal{O}_{N_{2,1}})$ (see [1, Section 2] and [4, Section 4] for details).

The following fact is well-known (cf. [5, Section 5.4] and [1, Proposition 2.2.2]).

Proposition 3.4. *For $F = \Sigma_{1,1}$ or $N_{2,1}$, the following hold.*

1. *The restriction of any type-preserving* $\mathrm{PSL}(2,\mathbb{C})$*- representation of $\pi_1(\mathcal{O}_F)$ to $\pi_1(F)$ is type-preserving.*
2. *Conversely, every type-preserving* $\mathrm{PSL}(2,\mathbb{C})$*-representation of $\pi_1(F)$ extends to a unique type-preserving* $\mathrm{PSL}(2,\mathbb{C})$*-representation of $\pi_1(\mathcal{O}_F)$.*

By this proposition, the following are well-defined.

Definition 3.5. (1) For $F = \Sigma_{1,1}$ or $\mathcal{O}_{\Sigma_{1,1}}$, the symbol $\Omega(\Sigma_{1,1})$ denotes the space of all type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representations ρ_1 of $\pi_1(F)$.

(2) For $F = N_{2,1}$ or $\mathcal{O}_{N_{2,1}}$, the symbol $\Omega(N_{2,1})$ (resp. $\Omega'(N_{2,1})$) denotes the space of all type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representations ρ_2 of $\pi_1(F)$ such that $\mathrm{tr}(\rho_2(K_{N_{2,1}})) = -2$ (resp. $\mathrm{tr}(\rho_2(K_{N_{2,1}})) = +2$).

Remark 3.6. For any $\rho_2 \in \Omega'(N_{2,1})$, the isometries $\rho_2(Q_0Q_2) = \rho_2(Y_2)$ and $\rho_2(K_{N_{2,1}})$ have a common fixed point (see [3, Lemma 4.5(ii)]), and hence ρ_2 is indiscrete or non-faithful (see [3, Lemma 4.7]).

Definition 3.7. (1) Let ρ_1 be an element of $\Omega(\Sigma_{1,1})$. Fix a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Set

$$(x_1, x_{12}, x_2) = (\mathrm{tr}(\rho_1(X_1)), \mathrm{tr}(\rho_1(X_1X_2)), \mathrm{tr}(\rho_1(X_2))),$$

where $X_1 = P_2P_1$ and $X_2 = P_0P_1$. Suppose that $x_1x_{12}x_2 \neq 0$. Then we call the following triple $(a_0, a_1, a_2) \in (\mathbb{C}^*)^3$ the *complex probability* associated with $\{\rho_1(P_j)\}$, where $\mathbb{C}^* =$

$\mathbb{C}-\{0\}$.

$$a_0=\frac{x_1}{x_{12}x_2},\ a_1=\frac{x_{12}}{x_2x_1},\ a_2=\frac{x_2}{x_1x_{12}}.$$

(2) Let ρ_2 be an element of $\Omega(N_{2,1})$. Fix a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$. Set

$$(y_1,y_{12},y_2)=(\mathrm{tr}(\rho_2(Y_1))/i,\mathrm{tr}(\rho_2(Y_1Y_2))/i,\mathrm{tr}(\rho_2(Y_2))),$$

where $Y_1=Q_0Q_1$ and $Y_2=Q_0Q_2$. Set $y'_{12}=\mathrm{tr}(\rho_2(Y_1Y_2^{-1}))/i=y_1y_2-y_{12}$. Suppose that $y_1y_2y'_{12}\neq 0$. Then we call the following triple $(b_0,b_1,b_2)\in(\mathbb{C}^*)^3$ the *complex probability associated with* $\{\rho_2(Q_j)\}$.

$$b_0+b_1+b_2=1,\ \text{where } b_0=\frac{y_1}{y_2y'_{12}},\ b_1=\frac{4}{y_1y_2y'_{12}},\ b_2=\frac{y'_{12}}{y_1y_2}.$$

Remark 3.8. (1) For any sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and any $\rho_1\in\Omega(\Sigma_{1,1})$, the complex probability (a_0,a_1,a_2) associated with $\{\rho_1(P_j)\}$ satisfies the following identity (see [1, Lemma 2.4.1(1)] for details):

$$a_0+a_1+a_2=1.$$

(2) For any sequence of elliptic generators $\{Q_k\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ and any $\rho_2\in\Omega(N_{2,1})$, the complex probability (b_0,b_1,b_2) associated with $\{\rho_2(Q_j)\}$ satisfies the following identity (see [4, Section 4] for details):

$$b_0+b_1+b_2=1.$$

We introduce the following proposition (cf. [1, Proposition 2.4.4] and [4, Propositions 4.8 and 4.11]).

Proposition 3.9. (1) *For any triple $(a_0,a_1,a_2)\in(\mathbb{C}^*)^3$ such that $a_0+a_1+a_2=1$ and for any sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$, there is an element $\rho_1\in\Omega(\Sigma_{1,1})$ such that the complex probability associated with $\{\rho_1(P_j)\}$ is equal to (a_0,a_1,a_2).*

(2) *For any triple $(b_0,b_1,b_2)\in(\mathbb{C}^*)^3$ such that $b_0+b_1+b_2=1$ and for any sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$, there is an element $\rho_2\in\Omega(N_{2,1})$ such that the complex probability associated with $\{\rho_2(Q_j)\}$ is equal to (b_0,b_1,b_2).*

Notation 3.10. (1) Let ρ_1 be an element of $\Omega(\Sigma_{1,1})$ and let $\{P_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Let ξ be the automorphism of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ given by the following (cf. [1, Proposition 2.1.6] and [4, Proposition 3.3]):

$$(\xi(P_0),\xi(P_1),\xi(P_2))=(P_2^{P_1},P_1,P_0^K).$$

If the complex probability associated with $\{\rho_1(\xi^k(P_j))\}$ is well-defined, then we denote it by $(a_0^{(k)},a_1^{(k)},a_2^{(k)})$.

(2) Let ρ_2 be an element of $\Omega(N_{2,1})$ and let $\{Q_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{N_{2,1}})$. Let σ be the automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ given by Proposition 2.2(1.2), namely,

$$(\sigma(Q_0),\sigma(Q_1),\sigma(Q_2))=(Q_2,Q_1^{Q_2Q_0},Q_0^{Q_2}).$$

If the complex probability associated with $\{\rho_2(\sigma^k(Q_j))\}$ is well-defined, then we denote it by $(b_0^{(k)},b_1^{(k)},b_2^{(k)})$.

The following lemma can be verified by simple calculation (cf. [1, Lemma 2.4.1] and [4, Lemmas 4.10 and 4.13]).

Lemma 3.11. (1) *Let ρ_1 be an element of $\Omega(\Sigma_{1,1})$ and let $\{P_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Suppose that the complex probability $(a_0^{(k)}, a_1^{(k)}, a_2^{(k)})$ associated with $\{\rho_1(\xi^k(P_j))\}$ is well-defined for any $k \in \mathbb{Z}$. Then we have the following identities (cf. Figure 4):*

$$a_0^{(k+1)} = 1 - a_2^{(k)},\ a_1^{(k+1)} = \frac{a_1^{(k)} a_2^{(k)}}{1 - a_2^{(k)}},\ a_2^{(k+1)} = \frac{a_2^{(k)} a_0^{(k)}}{1 - a_2^{(k)}},$$

$$a_0^{(k-1)} = \frac{a_2^{(k)} a_0^{(k)}}{1 - a_0^{(k)}},\ a_1^{(k-1)} = \frac{a_0^{(k)} a_1^{(k)}}{1 - a_0^{(k)}},\ a_2^{(k-1)} = 1 - a_0^{(k)}.$$

(2) *Let ρ_2 be an element of $\Omega(N_{2,1})$ and let $\{Q_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{N_{2,1}})$. Suppose that the complex probability $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$ associated with $\{\rho_2(\sigma^k(Q_j))\}$ is well-defined for any $k \in \mathbb{Z}$. Then we have the following identities (cf. Figure 5):*

$$b_0^{(k+1)} = 1 - b_2^{(k)},\ b_1^{(k+1)} = \frac{b_1^{(k)} b_2^{(k)}}{1 - b_2^{(k)}},\ b_2^{(k+1)} = \frac{b_2^{(k)} b_0^{(k)}}{1 - b_2^{(k)}},$$

$$b_0^{(k-1)} = \frac{b_2^{(k)} b_0^{(k)}}{1 - b_0^{(k)}},\ b_1^{(k-1)} = \frac{b_0^{(k)} b_1^{(k)}}{1 - b_0^{(k)}},\ b_2^{(k-1)} = 1 - b_0^{(k)}.$$

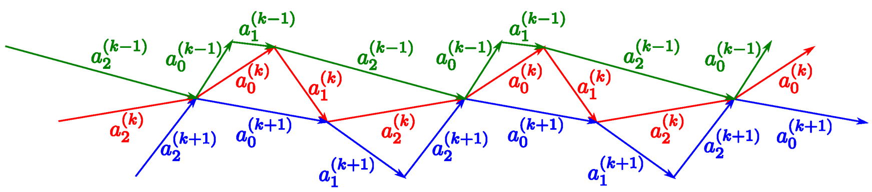

FIGURE 4. Adjacent complex probabilities of $\rho_1 \in \Omega(\Sigma_{1,1})$

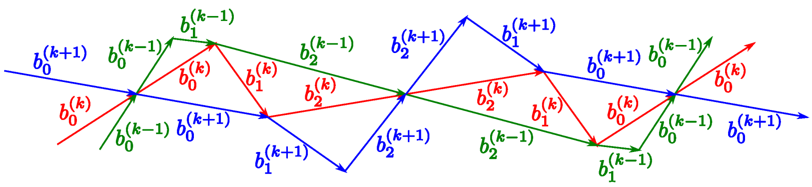

FIGURE 5. Adjacent complex probabilities of $\rho_2 \in \Omega(N_{2,1})$

Throughout this paper, we employ the following convention.

Convention 3.12. (1) For any element $\rho_1 \in \Omega(\Sigma_{1,1})$, after taking conjugate of ρ_1 by some element of $\mathrm{PSL}(2,\mathbb{C})$, we always assume that ρ_1 is normalized so that the following identity is satisfied.

$$\rho_1(K) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(2) For any element $\rho_2 \in \Omega(N_{2,1})$, after taking conjugate of ρ_2 by some element of $\mathrm{PSL}(2,\mathbb{C})$, we always assume that ρ_2 is normalized so that the following identities are satisfied.

$$\rho_2(K_0) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \rho_2(K_0) = \begin{pmatrix} i & -2i \\ 0 & -i \end{pmatrix}.$$

Now we give a partial answer to Problem 3.3. By [4, Lemma 4.15], we may only consider the problem for the quotient orbifolds. Our partial answer to the commensurability problem for representations of the fundamental groups of the orbifolds $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{N_{2,1}}$ is given as follows.

Theorem 3.13. *Under Convention 3.12, the following hold:*

(1) *Let ρ_1 be an element of $\Omega(\Sigma_{1,1})$. Suppose that ρ_1 is faithful. Then the following conditions are equivalent.*

(i) *There exists a faithful representation $\rho_2 \in \Omega(N_{2,1})$ which is commensurable with ρ_1.*

(ii) *There exist a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and an integer k_0 such that the complex probability (a_0, a_1, a_2) associated with $\{\rho_1(P_j)\}$ satisfies the following identity under Notation 3.10(1) (cf. Figure 6):*

$$(a_0^{(k_0)}, a_1^{(k_0)}, a_2^{(k_0)}) = (a_2, a_1, a_0).$$

(iii) *There exists a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ such that the complex probability (a_0, a_1, a_2) associated with $\{\rho_2(P_j)\}$ satisfies one of the following identities:*

(α) $(a_0^{(0)}, a_1^{(0)}, a_2^{(0)}) = (a_2, a_1, a_0)$,

(β) $(a_0^{(1)}, a_1^{(1)}, a_2^{(1)}) = (a_2, a_1, a_0)$.

(2) *If the conditions in (1) hold, the representation ρ_2 is unique up to precomposition by an automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ preserving $K_{N_{2,1}}$.*

(3) *Moreover, the following hold:*

(α) *ρ_1 extends to a type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\mathcal{O}_\alpha)$ if and only if ρ_1 satisfies the condition (iii)-(α). Moreover, if these conditions are satisfied, the extension is unique.*

(β) *ρ_1 extends to a type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\mathcal{O}_\beta)$ if and only if ρ_1 satisfies the condition (iii)-(β). Moreover, if these conditions are satisfied, the extension is unique.*

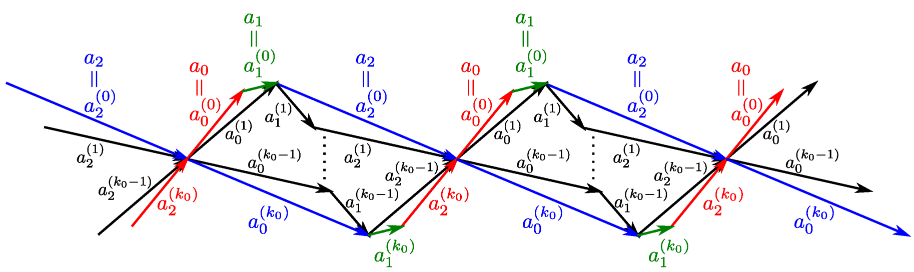

FIGURE 6. $(a_0^{(k_0)}, a_1^{(k_0)}, a_2^{(k_0)}) = (a_2, a_1, a_0)$

Proof. We only show the implication (1)-(i) $\Rightarrow$ (1)-(ii) and the assertion (2) because the other assertions can be proved by an argument similar to [4, Theorem 5.1].

We first prove the implication (1)-(i) $\Rightarrow$ (1)-(ii). Suppose that there exists a faithful representation $\rho_2 \in \Omega(N_{2,1})$ which is commensurable with ρ_1. Then, by [4, Theorem 5.1(1)-(ii)], there exist a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ and an integer k_0 such that the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ satisfies the following identity under Notation 3.10(2):

$$(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0).$$

By Proposition 3.9(1), there is an element $\rho_1' \in \Omega(\Sigma_{1,1})$ such that the complex probability (a_0', a_1', a_2') associated with $\{\rho_1'(P_j')\}$ is equal to (b_0, b_1, b_2) for some sequence of elliptic generators $\{P_j'\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Moreover, we can prove that ρ_1' and ρ_2 are commensurable (see proof of the implication (1)-(ii) $\Rightarrow$ (1)-(i) in [4, Theorem 5.1] for details). Hence, by [4, Theorem 5.1(2)], there is an automorphism f of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ preserving K such that $\rho_1 \circ f = \rho_1'$. Set $\{P_j\} = \{f(P_j')\}$. Then $\{P_j\}$ is also a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and $\rho_1(P_j) = \rho_1'(P_j')$. Hence the complex probability (a_0, a_1, a_2) associated with $\{\rho_1(P_j)\}$ is equal to $(a_0', a_1', a_2') = (b_0, b_1, b_2)$. By Lemma 3.11, the complex probability $(a_0^{(k_0)}, a_1^{(k_0)}, a_2^{(k_0)})$ associated with $\{\rho_1(\xi^{k_0}(P_j))\}$ is equal to the complex probability $(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)})$ associated with $\{\rho_2(\sigma^k(Q_j))\}$. Hence we have

$$(a_0^{(k_0)}, a_1^{(k_0)}, a_2^{(k_0)}) = (b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0) = (a_2, a_1, a_0).$$

Next we prove the assertion (2). Let ρ_2 and ρ_2' be elements of $\Omega(N_{2,1})$ such that they are commensurable with ρ_1. Then there exist double coverings $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ and $p_2, p_2' : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ such that $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$ and $\rho_1 \circ (p_1)_* = \rho_2' \circ (p_2')_*$. Pick an elliptic generator triple (Q_0, Q_1, Q_2) of $\pi_1(\mathcal{O}_{N_{2,1}})$. Note that there is a unique covering from $\mathcal{O}_{\Sigma_{1,2}}$ to $\mathcal{O}_{N_{2,1}}$ up to equivalence which corresponds to the epimorphism $\phi_2 : \pi_1(\mathcal{O}_{N_{2,1}}) \to \mathbb{Z}/2\mathbb{Z}$ defined by the following formula (see [4, Section 2] for details):

$$\phi_2(Q_j) = \begin{cases} 0 \text{ if } j = 0 \text{ or } 2, \\ 1 \text{ if } j = 1. \end{cases}$$

Hence there is a self-homeomorphism g of $\mathcal{O}_{\Sigma_{1,2}}$ such that $p_2' = g \circ p_2$, and $Q_0, Q_2 \in (p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))$. Set $Q_0' = (p_2')_* \circ (p_2)_*^{-1}(Q_0)$ and $Q_2' = (p_2')_* \circ (p_2)_*^{-1}(Q_2)$.

Claim 3.14. *(Q_0', Q_1, Q_2') is also an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$.*

Proof. Note that Q_0' and Q_2' have order 2, because

(1) $(p_2')_* \circ (p_2)_*^{-1} : (p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}})) \to (p_2')_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))$ is an isomorphism and

(2) Q_0 and Q_2 have order 2.

Since $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$ and $\rho_1 \circ (p_1)_* = \rho_2' \circ (p_2')_*$, we have $\rho_2 \circ (p_2)_* = \rho_2' \circ (p_2')_*$. Hence we have

$$\begin{aligned} \rho_2(Q_0) &= \rho_2 \circ (p_2)_*((p_2)_*^{-1}(Q_0)) \\ &= \rho_2' \circ (p_2')_*((p_2)_*^{-1}(Q_0)) && \text{by } \rho_2 \circ (p_2)_* = \rho_2' \circ (p_2')_* \\ &= \rho_2'(Q_0') && \text{by } Q_0' = (p_2')_* \circ (p_2)_*^{-1}(Q_0). \end{aligned}$$

Similarly, we have $\rho_2(Q_2) = \rho'_2(Q'_2)$. Hence we have

$$\begin{aligned}
\rho'_2(Q_1^{Q'_2} Q_1^{Q'_0}) &= \rho_2(Q_1^{Q_2} Q_1^{Q_0}) && \text{by } \rho_2(Q_j) = \rho'_2(Q'_j) \text{ for } j = 0, 2 \\
&= \rho_2(K_{N_{2,1}}) && \text{by } Q_1^{Q_2} Q_1^{Q_0} = K_{N_{2,1}} \\
&= \rho'_2(K_{N_{2,1}}) && \text{by Convention 3.12.}
\end{aligned}$$

Since ρ'_2 is faithful, we have $Q_1^{Q'_2} Q_1^{Q'_0} = K_{N_{2,1}}$. Thus, by Remark 2.6, the triple (Q'_0, Q_1, Q'_2) is an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$. □

By this claim, there are elliptic generator triples (Q_0, Q_1, Q_2) and (Q'_0, Q_1, Q'_2) of $\pi_1(\mathcal{O}_{N_{2,1}})$ satisfying the following identity:

$$(\rho_2(Q_0), \rho_2(Q_1), \rho_2(Q_2)) = (\rho'_2(Q'_0), \rho'_2(Q_1), \rho'_2(Q'_2)).$$

By Proposition 2.2(2), there is an automorphism f of $\pi_1(\mathcal{O}_{N_{2,1}})$ preserving $K_{N_{2,1}}$ such that f maps (Q_0, Q_1, Q_2) to (Q'_0, Q_1, Q'_2). Hence we have $\rho_2 = \rho'_2 \circ f$. □

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